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ANALYSIS OF THE PROBLEM OF THERMAL FLAME PROPAGATION BY THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

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V.S.BERMAN and Iu.S.RIAZANTSEV
(Moscow)

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The method of matched asymptotic expansions is used to obtain two approximations for the rate of steady thermal propagation of a flame front through a homogeneous, gaseous fuel mixture. The coefficients of heat conductivity and diffusion, and the density of the medium, are assumed to be functions of the temperature and concentration of the reacting substance in the gas. An analytic relationship between the flame velocity and the gradients of these functions at the hot boundary of the combustion zone is established. The formulas derived in [1, 2] represent particular cases of the expression obtained in the present paper for the flame velocity.

1. Equations and boundary conditions. In the coordinate system moving with the velocity of the flame, the equations of the steady thermal flame propagation in a homogeneous gaseous mixture can be written, under a number of simplifying assumptions, in the form

$$\frac{d}{dx}\left(\frac{\lambda}{c}\frac{dT}{dx}\right) - m\frac{dT}{dx} + \frac{h}{c}a^n\rho^n\Phi(T) = 0 \qquad (1.1)$$

$$\frac{d}{dx}\left(\rho D\frac{da}{dx}\right) - m\frac{da}{dx} - a^n\rho^n\Phi(T) = 0 \qquad (1.2)$$

$$x = -\infty, \quad T = T_-, \quad a = a_-; \quad x = \infty, \quad \frac{dT}{dx} = a = 0 \qquad (1.3)$$

Here T is the temperature, a is the concentration of the reacting substance, m is the mass rate of flame propagation and also the eigenvalue of the problem (1.1) – (1.3), c is a constant representing heat capacity of the gas, $\rho = \rho(T, a)$ is the gas density, n is the order of reaction, $D = D(T, a)$ is the diffusion coefficient, $\lambda = \lambda(T, a)$ is the heat conductivity coefficient, $\Phi(T)$ expresses the temperature dependence of the rate of chemical reaction, T_- is the initial temperature of the gas, a_- is the initial concentration and $h = \text{const}$ is the heat effect of the reaction.

The system (1.1) and (1.2) has the following first integral

$$\frac{\lambda}{c}\frac{dT}{dx} + \frac{h}{c}\rho D\frac{da}{dx} - m\left(T + \frac{ha}{c}\right) = \text{const} \qquad (1.4)$$

From (1.4) and from the conditions $(dT / dx)_{\pm\infty} = (da / dx)_{\pm\infty} = 0$ which follow from (1.2), we find

$$\frac{h}{c}\rho D\frac{da}{dx} = m\left(T + \frac{ha}{c} - T_+\right) - \frac{\lambda}{c}\frac{dT}{dx}, \quad T_+ = T_- + \frac{ha_-}{c} \qquad (1.5)$$

The minus and plus subscripts denote the quantity at the cold and hot boundary of the zone of combustion.

Let us introduce the following dimensionless variables in the Eqs. (1.1) and (1.5) equivalent to the system (1.1), (1.2):

$$\tau = (T - T_-) / (T_+ - T_-), \qquad y = (a_- - a) / a_-$$

and pass from the independent variable x to the independent variable τ, and to the new unknown function

$$\mathrm{P} = \frac{\lambda}{c} \frac{d\tau}{dx}$$

Then (1.1), (1.5) and (1.3) will be replaced by

$$\mathrm{P} \frac{d\mathrm{P}}{d\tau} - m\mathrm{P} + \frac{\lambda}{c} \rho^n a_-^{n-1} (1 - y)^n \Phi(\tau) = 0 \tag{1.6}$$

$$\frac{dy}{d\tau} = LNm \frac{y - \tau}{\mathrm{P}} + LN, \quad L = \frac{\lambda_+}{D_+\rho_+ c}, \qquad N(\tau, y) = \frac{\lambda}{DL\rho c} \tag{1.7}$$

$$\tau = 0, \quad \mathrm{P} = 0, \quad y = 0; \quad \tau = 1, \quad \mathrm{P} = 0, \quad y = 1 \tag{1.8}$$

Let us specify the form of the function $\Phi(T)$, assuming that the dependence of the rate of chemical reaction of temperature obeys the Arrhenius law

$$\Phi(T) = A \exp \frac{-E}{RT} \tag{1.9}$$

Here E is the activation energy, R is the gas constant and A is a frequency factor. Taking (1.9) into account we can write (1.6) in the form

$$\mathrm{P} \frac{d\mathrm{P}}{d\tau} - m\mathrm{P} + e^{-\beta} K(\tau, y)(1 - y)^n \exp \frac{-\beta(1-\tau)}{\tau + \sigma} = 0 \tag{1.10}$$

$$K(\tau, y) = \frac{a_-^{n-1}\rho}{c} \lambda A, \qquad \sigma = \frac{T_- c}{h a_-}, \qquad \beta = \frac{E}{RT_+}$$

Let

$$q = \mathrm{P} \exp \beta/2, \qquad M = m \exp \beta/2$$

Then the problem (1.6) – (1.8) assumes the form

$$q \frac{dq}{d\tau} - Mq + K(\tau, y)(1 - y)^n \exp \frac{-\beta(1-\tau)}{\tau + \sigma} = 0 \tag{1.11}$$

$$\frac{dy}{d\tau} = LN(\tau, y) M \frac{y - \tau}{q} + LN(\tau, y) \tag{1.12}$$

$$\tau = 0, \qquad q = y = 0 \tag{1.13}$$

$$\tau = 1, \qquad q = 0, \qquad y = 1 \tag{1.14}$$

We note that the boundary value problem written in the form (1.11) – (1.14) has no solution, since the function (1.9) does not vanish when $T = T_-$ (so-called difficulty connected with the cold boundary [1, 3]). In the theory of thermal flame propagation it is usually assumed that the dependence of the rate of combustion on temperature has the form (1.9) everywhere except in a certain temperature interval $T_0 \leqslant T \leqslant T^\circ < T_+$, in which $\Phi(T) = 0$. This ensures the existence of a solution of the problem (1.11) – (1.14) [1, 4]. In the present paper Eqs. (1.11) and (1.12) are used in an approximate form such, that the difficulty connected with the cold boundary is automatically overcome.

2. Method of solution. Equation (1.11) contains a dimensionless parameter β, which is usually much larger than unity. Its typical values are $\beta \approx 10$. This makes possible to obtain an approximate solution of the problem by the method of matching asymptotic expansions [5]. The form of Eqs. (1.11) and (1.12) implies that the interval of variation $0 \leqslant \tau \leqslant 1$ of the variable can be split into two sub-intervals. In the first sub-interval adjacent to $\tau = 0$ and constituting the major portion of the interval (the outer region) the third term in (1.11) is substantially smaller than the remaining terms since it contains an exponentially decreasing factor, the exponential index being of the order of β. In the second sub-interval adjacent to $\tau = 1$ (the inner region) large values of β in the exponential index are compensated by the smallness of the quantity $(1 - \tau)$, therefore the third term in (1.11) becomes significant. In order to determine the order of magnitude of each term of (1.11) and (1.12) in the inner region, we introduce the variable $\tau_* = \beta(1 - \tau)$. Then (1.11), (1.12) and (1.14) become

$$q\frac{dq}{d\tau} + \beta^{-1}Mq - \beta^{-1}(1-y)^n K(1-\beta^{-1}\tau_*, y)\exp\frac{-\tau_*}{\sigma+1-\beta^{-1}\tau_*} = 0 \quad (2.1)$$

$$\frac{dy}{d\tau_*} = \beta^{-1}LN(1-\beta^{-1}\tau_*, y)\left[\frac{M}{q}(1-\beta^{-1}\tau_* - y) - 1\right] \quad (2.2)$$

$$\tau_* = 0, \quad q = 0, \quad y = 1 \quad (2.3)$$

We shall seek a solution of the problem in the outer and inner region in the form of expansions in powers of a small parameter β^{-1}.

In the inner region we have

$$q_*(\tau_*) = G_0(\beta)q_0(\tau_*) + G_1(\beta)q_1(\tau_*) + \ldots \quad (2.4)$$
$$y_*(\tau_*) = F_0(\beta)y_0(\tau_*) + F_1(\beta)y_1(\tau_*) + F_2(\beta)q_2(\tau_*) + \ldots$$

and in the outer region

$$q^*(\tau) = g_0(\beta)q^{(0)}(\tau) + g_1(\beta)q^{(1)}(\tau) + \ldots \quad (2.5)$$
$$y^*(\tau) = f_0(\beta)y^{(0)}(\tau) + f_1(\beta)y^{(1)}(\tau) + f_2(\beta)y^{(2)}(\tau) + \ldots$$

The expansions for the eigenvalue M is the same in both zones

$$M = \alpha_0(\beta)M_0 + \alpha_1(\beta)M_1 + \ldots \quad (2.6)$$

The coefficients dependent on β in the expansions (2.4) – (2.6) must satisfy the following conditions for $\beta \to \infty$:

$$\frac{G_{i+1}}{G_i} \to 0, \quad \frac{F_{i+1}}{F_i} \to 0, \quad \frac{g_{i+1}}{g_i} \to 0, \quad \frac{f_{i+1}}{f_i} \to 0, \quad \frac{\alpha_{i+1}}{\alpha_i} \to 0 \quad (i = 0, 1, 2, \ldots) \quad (2.7)$$

The functions $q_i(\tau_*)$ and $y_i(\tau_*)$ are determined successively from Eqs. (2.1), (2.2) and boundary condition (2.3), while the functions $q^{(i)}(\tau)$ and $y^{(i)}(\tau)$ are obtained from (1.11), (1.12) and boundary condition (1.13). The still undetermined arbitrary constants and terms of the series (2.6) defining the eigenvalue M of the problem are found from the condition of matching the inner (2.4) and outer (2.5) expansions. The matching procedure consists of requiring that the corresponding terms of the asymptotic expansions for $q_*(\tau_*)$ and $y_*(\tau_*)$ with $\tau_* \to \infty$ and for $q^*(\tau)$ and $y^*(\tau)$ with $\tau \to 1$, coincide. The form of the coefficients G_i, F_i, g_i, f_i and α_i is determined from the boundary and the matching conditions.

In the course of analysis it is assumed that the functions $K(\tau, y)$ and $N(\tau, y)$ as well as their derivatives with respect to τ and y are continuous and bounded functions and that they are of the order of unity, as are L and n.

3. Zeroth order approximation for the flame propagation velocity. Let us insert the expansions (2.5) and (2.6) into (1.11). Since the relations

$$\exp(-\beta)\,/\,f_i(\beta) \to 0, \qquad \exp(-\beta)\,/\,g_i(\beta) \to 0,$$
$$\exp(-\beta)\,/\,\alpha_i(\beta) \to 0$$

hold for $\beta \to \infty$, the approximate equation for q in the outer region can be written in the form

$$q^* \, dq^*/d\tau - Mq^* = 0 \tag{3.1}$$

Out of the two solutions of (3.1) satisfying the condition (1.13) we choose, in accordance with the physical sense of the problem, the solution

$$q^* = M\tau \tag{3.2}$$

From (3.2) it follows that

$$\alpha_0(\beta) = g_0(\beta), \qquad \alpha_1(\beta) = g_1(\beta), \ldots \tag{3.3}$$
$$q^{(0)}(\tau) = M_0\tau, \qquad q^{(1)}(\tau) = M_1\tau, \ldots$$

Inserting (3.2) into (1.12) we obtain the approximate equation for the function $y(\tau)$ in the outer region

$$\frac{dy^*}{d\tau} = LN(\tau, y^*)\frac{y^*}{\tau} \tag{3.4}$$

Substituting (2.5) into (3.4) and discarding the terms of higher order than $f_0(\beta)$, we obtain the following equation for $y^{(0)}(\tau)$:

$$\frac{dy^{(0)}}{d\tau} = LN(\tau, f_0 y^{(0)})\frac{y^{(0)}}{\tau} \tag{3.5}$$

Equation (3.5) defines, in the zeroth order approximation, the distribution of concentration near the cold boundary of the zones of combustion and has bounded solutions $y^{(0)}(\tau)$ in the interval $0 \leqslant \tau < 1$. Since the point $\tau = 0, y = 0$ is a node type singularity, the boundary condition (1.13) is insufficient for selecting the unique solution of (3.5).

Using (3.3) and (3.5), we can write the single-term outer expansions of $g_0(\beta)\,q^{(0)}(\tau)$ and $f_0(\beta)\,y^{(0)}(\tau)$ with $\tau \to 1$ as functions of the inner variable τ_*

$$g_0 q^{(0)}(\tau) = g_0 M_0 \tau = g_0 M_0 - \beta^{-1} g_0 M_0 \tau_* \tag{3.6}$$
$$f_0 y^{(0)}(\tau) = f_0 y^{(0)}(1) - \beta^{-1} f_0 L y_{(1)}^{(0)} \tau_* + \ldots$$

The matching condition (3.6) and the inner expansions lead us to the conclusion that

$$G_0(\beta) = g_0(\beta) = \alpha_0(\beta), \quad F_0(\beta) = f_0(\beta), \quad G_1(\beta) = \beta^{-1} g_0(\beta), \quad F_1(\beta) = \beta^{-1} f_0(\beta) \tag{3.7}$$

The boundary condition (2.3) implies that

$$F_0(\beta) = 1 \tag{3.8}$$

Inserting the expansions (2.4) into (2.2) and collecting the terms of like order of smallness we obtain, with (3.7) and (3.8) taken into accont,

$$dy_0/d\tau_* = 0, \quad dy_1/d\tau_* = -L \tag{3.9}$$

The solutions of (3.9) satisfying the condition (2.3) have the form

$$y_0(\tau_*) = 1, \quad y_1(\tau_*) = -L\tau_* \tag{3.10}$$

Substituting the expansions (2.4) into (2.1) and taking (3.7), (3.8) and (3.10) into account, we obtain the following equation for the terms of the smallest order in β^{-1}:

$$G_0^2(\beta)\, q_0 \frac{dq_0}{d\tau_*} = \beta^{-(n+1)} L^n K_+ \tau_*^{\,n} \exp\frac{-\tau_*}{\sigma+1} \tag{3.11}$$

From (3.11) it follows that

$$G_0(\beta) = \beta^{-n+1/2} \tag{3.12}$$

A solution of (3.11) satisfying the boundary condition (2.3) has the form

$$q_0^2(\tau_*) = 2L^n K_+ (\sigma+1)^{n+1} \gamma\left(n+1, \frac{\tau_*}{\sigma+1}\right) \tag{3.13}$$

$$\gamma(n, z) = \Gamma(n) - \Gamma(n, z), \qquad \Gamma(n, z) = \int_z^\infty t^{n-1} e^{-t}\, dt$$

$$\Gamma(n, 0) \equiv \Gamma(n)$$

The condition of matching the zero order terms of the inner and outer expansion of $q(\tau)$, the terms defined by the formulas (3.3) and (3.13), respectively, yields the zeroth term of the expansion for the eigenvalue M of the problem

$$M_0^2 = 2L^n K_+ (\sigma+1)^{n+1} \Gamma(n+1) \tag{3.14}$$

In the dimensional variables the zeroth order approximation for the mass propagation of the combustion front has the form

$$m_0 = \left[2L^n \Gamma(n+1) \frac{\lambda}{c} \rho_+^n a_-^{n-1} A \left(\frac{T_+}{T_+ - T_-}\right)^{n+1} \left(\frac{E}{RT_+}\right)^{-(n+1)} \exp\frac{-E}{RT_+}\right]^{1/2} \tag{3.15}$$

When $L = 1$, the formula (3.15) becomes identical with the formula for the velocity of combustion established in [1].

4. First order approximation for the flame propagation velocity. To determine the second term in the expansion (2.6) we must find the coefficients $\alpha_1(\beta) = g_1(\beta)$, $f_1(\beta)$, $G_1(\beta)$, $g_1(\beta)$, $F_2(\beta)$ and functions $q_1(\tau)$, $q^{(1)}(\tau_*)$ $y^{(1)}(\tau)$ and $y_2(\tau_*)$. From (3.7) and (3.12) it follows that

$$G_1(\beta) = \beta^{-(n+3)/2} \tag{4.1}$$

The two-term inner expansion of $y(\tau_*)$ has, in accordance with (3.10), the form

$$y(\tau_*) = 1 - \beta^{-1} L \tau_* \tag{4.2}$$

The function (4.2) matches completely the two-term expansion of the function $y^{(0)}(\tau)$ given by the formula (3.6), provided that we take (3.7) and (3.8) into account and set $y^{(0)}(1) = 1$. From the condition of matching (4.2) and $y^{(0)}(\tau)$ it follows that the expansion for $y(\tau)$ in the outer region contains no terms of the order of β^{-1}. We should therefore set

$$f_1(\beta) = \beta^{-2}, \qquad y(\tau) = y^{(0)}(\tau) + \beta^{-2} y^{(1)}(\tau) \tag{4.3}$$

When $\tau \to 1$, the expansion (4.3) written as a function of the inner variable τ_* has the form

$$y^*(\tau) = 1 - \beta^{-1} L\tau_* + \beta^{-2} L\left[\left(\frac{\partial N}{\partial \tau}\right)_+ + L\left(\frac{\partial N}{\partial y}\right)_+ + L - 1\right]\frac{\tau_*^2}{2} + \\ + \beta^{-2} y^{(1)}(1) + \ldots \tag{4.4}$$

The quantity $y^{(1)}(1)$ appearing here must be determined in the course of matching the outer and inner expansions analogously to $y^{(0)}(1)$ in (3.6). In accordance with (4.4) we set

$$F_2(\beta) = \beta^{-2} \tag{4.5}$$

Inserting the expansions (2.4) into (2.2), taking (3.7), (3.8), (3.10), (3.12 – 3.14) and (4.5) into account and collecting the terms of the order of β^{-2} we obtain

$$\frac{dy_2}{d\tau_*} = L(L-1)\Gamma^{1/2}(n+1)\gamma^{-1/2}\left(n+1, \frac{\tau_*}{\sigma+1}\right)\tau_* +$$

$$+L\left[\left(\frac{\partial N}{\partial \tau}\right)_+ + L\left(\frac{\partial N}{\partial y}\right)_+\right]\frac{\tau_*^2}{2} \qquad (4.6)$$

Solution of (4.6) satisfying the boundary condition (2.3) has the form

$$y_2(\tau_*) = L(L-1)(\sigma+1)^2\,\Gamma^{1/2}(n+1)\int\limits_0^{(\sigma+1)^{-1}\tau_*}\gamma^{-1/2}(n+1,z)\,z\,dz + $$
$$+L\left[\left(\frac{\partial N}{\partial \tau}\right)_+ + L\left(\frac{\partial N}{\partial y}\right)_+\right]\frac{\tau_*^2}{2} \qquad (4.7)$$

Using (4.7) we find from the condition of matching $y(\tau_*) = 1-\beta^{-1}L\tau_* + \beta^{-2}y_2(\tau_*)$ with (4.4), that

$$y^{(1)}(1) = L(L-1)(\sigma+1)^2\int\limits_0^\infty [\Gamma^{1/2}(n+1)\,\gamma^{-1/2}(n+1,z)-1]\,z\,dz \qquad (4.8)$$

The condition (4.8) defines, similarly to the condition $y^{(0)}(1) = 1$, the unique solution of the problem (3.4) and (1.13).

After substituting the expansions (2.4) into Eq. (2.1), selecting the terms of like order of smallness and taking into account the previously derived results for the function $q_1(\tau_*)$, we obtain

$$\frac{dq_1q_0}{d\tau_*} + M_0q_0 + L^n\tau_*^nK_+\left\{\left[\left(\frac{\partial \ln K}{\partial \tau}\right)_+ + L\left(\frac{\partial \ln K}{\partial y}\right)_+\right]\tau_* + \right.$$
$$\left. + \frac{\tau_*^2}{(\sigma+1)^2} + \frac{ny_2}{L\tau_*}\right\}e^{-\tau_*/(\sigma+1)} \qquad (4.9)$$

where the functions $q_0(\tau_*)$ and $y_2(\tau_*)$ have the form given by (3.13) and (4.7), respectively. Integrating (4.9) we obtain

$$q_1q_0 = -M_0^2(\sigma+1)\,\Gamma^{-1/2}(n+1)\int\limits_0^{(\sigma+1)^{-1}\tau_*}\gamma^{1/2}(n+1,z)\,dz -$$
$$-L^nK_+(\sigma+1)^{n+1}\gamma\left(n+3,\frac{\tau_*}{\sigma+1}\right) - L^nK_+\left[\left(\frac{\partial \ln K}{\partial \tau}\right)_+ + \right. \qquad (4.10)$$
$$\left. + L\left(\frac{\partial \ln K}{\partial y}\right)_+ + \frac{n}{2}\left(\frac{\partial N}{\partial \tau}\right)_+ + \frac{nL}{2}\left(\frac{\partial N}{\partial y}\right)_+\right](\sigma+1)^{n+2}\gamma\left(n+2\,\frac{\tau_*}{\sigma+1}\right) -$$
$$-L^nK_+n(L-1)(\sigma+1)^{n+2}\Gamma^{1/2}(n+1)\int\limits_0^{(\sigma+1)^{-1}\tau_*}\int\limits_0^{\tau}\gamma^{-1/2}(n+1,z)\,z\,dz\,t^{n-1}e^{-t}\,dt$$

The condition of matching the functions $q^{(0)}(\tau) + \beta^{-1}q^{(1)}(\tau) = M_0\tau + \beta^{-1}M_1\tau$ with $q_0(\tau_*) + \beta^{-1}q_1(\tau_*)$ yields

$$M_1 = \frac{M_0}{2}\left\{2(\sigma+1)J_1(n) - (n+2)(n+1) - (\sigma+1)(n+1)\left[\left(\frac{\partial \ln K}{\partial \tau}\right)_+ + \right.\right.$$
$$\left.\left. + L\left(\frac{\partial \ln K}{\partial y}\right)_+ + \frac{n}{2}\left(\frac{\partial N}{\partial \tau}\right)_+ + \frac{nL}{2}\left(\frac{\partial N}{\partial y}\right)_+\right] - (\sigma+1)(L-1)J_2(n)\right\} \qquad (4.11)$$

$$J_1(n) = \int\limits_0^\infty [1-\Gamma^{-1/2}(n+1)\,\gamma^{1/2}(n+1,z)]\,dz$$

$$J_2(n) = \frac{n}{\sqrt{\Gamma(n+1)}}\int\limits_0^\infty \frac{z\Gamma(n,z)\,dz}{\gamma^{1/2}(n+1,z)}$$

When $n = 1$ and $n = 2$, the integrals J_1 and J_2 are respectively:

$$J_1(1) = 1.344, \quad J_1(2) = 2.114; \quad J_2(1) = 2, \quad J_2(2) = 8.885$$

The two-term asymptotic formula for the mass flame front propagation velocity has the form

$$m = \beta^{-(n+1)/2}(M_0 + \beta^{-1}M_1)\exp\frac{-\beta}{2} \qquad (4.12)$$

In the case of $n = 1$, $K = \text{const}$ and $N = \text{const}$ considered in [2], we have

$$M_1 = M_0(\sigma + 1)J_1(1) - 3M_0 - (L - 1)M_0(\sigma + 1) \qquad (4.13)$$

It was shown in [2] that the two-term asymptotic expansion (4.13) for m gives good agreement with the results obtained by numerical methods for $\beta \geqslant 3$. The formulas (4.1), (3.14) and (4.13) yield a two-term asymptotic expansion for the combustion velocity for an arbitrary value of n and for a medium whose properties depend on the temperature and concentration.

From (3.14) and (4.11) it follows that the dependence of the properties of the medium on temperature and concentration is described by the manner in which the second term of the asymptotic expansion for the combustion velocity depends on the values of the gradients of the properties of the medium at the hot boundary of combustion.

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